

# An Elementary Proof of the Free-additivity of Voiculescu's Free Entropy

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**Abstract** D. Voiculescu [2] proved that a standard family of independent random unitary  $k \times k$  matrices and a constant  $k \times k$  unitary matrix is asymptotically free as  $k \rightarrow \infty$ . This result was a key ingredient in Voiculescu's proof [3] that his free entropy is additive when the variables are free. In this paper, we give a very elementary proof of a more detailed version of this result [2].  
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## 1 Preliminaries

The theory of free probability and free entropy was introduced by D. Voiculescu in the 1980's, and has become one of the most powerful and exciting new tools in the theory of von Neumann algebras. D. Voiculescu [2] proved that a standard family of independent random unitary  $k \times k$  matrices and a constant  $k \times k$  unitary matrix is asymptotically free as  $k \rightarrow \infty$ . To prove this result, Voiculescu used his noncommutative central limit theorem and the fact that the unitaries in the polar decomposition of a family of standard Gaussian random matrices form a standard family of independent unitary  $k \times k$  random matrices. In this paper, we will give a very elementary proof that uses only the basic properties of Haar measure and the definition of a unitary matrix.

Let  $\mathcal{M}_k(\mathbb{C})$  be the  $k \times k$  full matrix algebra with entries in  $\mathbb{C}$ , and  $\tau_k$  be the normalized trace on  $\mathcal{M}_k(\mathbb{C})$ , i.e.,  $\tau_k = \frac{1}{k}Tr$ , where  $Tr$  is the usual trace on  $\mathcal{M}_k(\mathbb{C})$ . Let  $\mathcal{U}_k$  be the group of all unitary matrices in  $\mathcal{M}_k(\mathbb{C})$ . For  $1 \leq i, j \leq k$ , define  $f_{i,j} : \mathcal{M}_k(\mathbb{C}) \rightarrow \mathbb{C}$  so that any element  $a$  in  $\mathcal{M}_k(\mathbb{C})$  is the matrix  $(f_{i,j}(a))$ ; i.e.,  $f_{ij}(a)$  is the  $(i, j)$ -entry of  $a$ .

An  $k \times k$  matrix  $u$  is unitary if and only if

- (1)  $\sum_{i=1}^k |f_{i,j_1}(u)|^2 = \sum_{j=1}^k |f_{i_1,j}(u)|^2 = 1$  for  $1 \leq i_1, j_1 \leq k$ , and
- (2)  $\sum_{i=1}^k f_{i,j_1}(u) \overline{f_{i,j_2}(u)} = \sum_{j=1}^k f_{i_1,j}(u) \overline{f_{i_2,j}(u)} = 0$ , whenever  $i_1 \neq i_2$  and  $j_1 \neq j_2$ .

Since  $\mathcal{U}_k$  is a compact group, there exists a unique normalized Haar measure  $\mu_k$  on  $\mathcal{U}_k$ . In addition,

$$\int_{\mathcal{U}_k} f(u) d\mu_k(u) = \int_{\mathcal{U}_k} f(vu) d\mu_k(u) = \int_{\mathcal{U}_k} f(uv) d\mu_k(u).$$

for every continuous function  $f : \mathcal{U}_k \rightarrow \mathbb{C}$  and  $v \in \mathcal{U}_k$ .

By the translation-invariance of  $\mu_k$ , we have the following lemmas (see also Lemma 12, Lemma 13 and Lemma 14 in [1]).

**Lemma 1** If  $g : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is a continuous function,  $\sigma$  and  $\rho$  are permutations of  $\{1, 2, \dots, k\}$ , then

$$\begin{aligned} & \int_{\mathcal{U}_k} g(f_{i_1 j_1}(u), f_{i_2 j_2}(u), \dots, f_{i_n j_n}(u)) d\mu_k(u) \\ &= \int_{\mathcal{U}_k} g(f_{\sigma(i_1), \rho(j_1)}(u), f_{\sigma(i_2), \rho(j_2)}(u), \dots, f_{\sigma(i_n), \rho(j_n)}(u)) d\mu_k(u). \end{aligned}$$

**Lemma 2** If  $\int_{\mathcal{U}_k} f_{i_1 j_1}(u) \cdots f_{i_m j_m}(u) \overline{f_{s_1 t_1}(u)} \cdots \overline{f_{s_r t_r}(u)} d\mu_k(u) \neq 0$ , then

- (1)  $m = r$ ,
- (2)  $(i_1, i_2, \dots, i_m)$  is a permutation of  $(s_1, s_2, \dots, s_r)$ ,
- (3)  $(j_1, j_2, \dots, j_m)$  is a permutation of  $(t_1, t_2, \dots, t_r)$ .

**Lemma 3** If  $d$  is the maximum cardinality of the sets  $\{i_1, \dots, i_n\}$ ,  $\{j_1, \dots, j_n\}$ ,  $\{s_1, \dots, s_r\}$  and  $\{t_1, \dots, t_r\}$ , then, for every positive integer  $k \geq d$ ,

$$\left| \int_{\mathcal{U}_k} f_{i_1 j_1}(u) \cdots f_{i_n j_n}(u) \overline{f_{s_1 t_1}(u)} \cdots \overline{f_{s_r t_r}(u)} d\mu_k(u) \right| \leq \frac{1}{P(k, d)},$$

where  $P(k, d) = k(k-1) \cdots (k-d+1)$ .

## 2 Main result

If  $f : \mathcal{F} \rightarrow \mathbb{C}$ , let  $\|f\|_\infty = \sup \{|f(x)| : x \in \mathcal{F}\}$ .

**Lemma 4** Let  $n, m, k$  be positive integers. Let  $F, G$  be finite subsets of  $\mathbb{N}$  with  $n = \text{Card}(F)$  and  $m = \text{Card}(G)$ . Suppose  $\{f_i, g_j : i \in F, j \in G\}$  is a family of mappings from  $\{1, \dots, k\}$  to  $\mathbb{C}$  such that  $\sum_{a=1}^k f_i(a) = 0$  for  $i \in F$ . Let  $H = \{1, \dots, k\}$ . Then

$$\left| \sum_{\sigma: F \cup G \xrightarrow{1-1} H} \prod_{i \in F} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \leq k^{m+\frac{n}{2}} (n+m)^n \prod_{i \in F} \|f_i\|_\infty \prod_{j \in G} \|g_j\|_\infty.$$

*Proof.* The proof is by induction on  $n$ . When  $n = 0$ , the obvious interpretation of the inequality is

$$\left| \sum_{\sigma} \prod_{j \in G} g_j(\sigma(j)) \right| \leq k^m \prod_{j \in G} \|g_j\|_\infty,$$

and it holds since the number of functions  $\sigma$  is no more than  $k^m$ .

Suppose the lemma holds for  $n$ . For  $n+1$ , let  $E$  be a subset of  $F$  with cardinality  $n$ , say it  $E = F \setminus \{b\}$ , where  $b \in F$ . Then we can define a one-to-one mapping  $\sigma : F \cup G \rightarrow H$  by defining the one-to-one mapping  $\sigma : E \cup G \rightarrow H$  and choosing  $s \notin \sigma(E \cup G)$  to be  $\sigma(b)$ . Then

$$\left| \sum_{\sigma: F \cup G \xrightarrow{1-1} H} \prod_{i \in F} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right|$$

$$\begin{aligned}
&= \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{s \notin \sigma(E \cup G)} f_b(s) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&= \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{s=1}^k f_b(s) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right. \\
&\quad \left. - \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{s \in \sigma(E \cup G)} f_b(s) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\quad \text{(using } \sum_{s=1}^k f_{n+1}(s) = 0 \text{)} \\
&= \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{s \in \sigma(E \cup G)} f_b(s) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&= \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{t \in E \cup G} f_b(\sigma(t)) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\leq \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{t \in E} f_b(\sigma(t)) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\quad + \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \sum_{t \in G} f_b(\sigma(t)) \right) \prod_{i \in E} f_i(\sigma(i)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\leq \sum_{t \in E} \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \prod_{i \in E, i \neq t} f_i(\sigma(i)) \right) (f_b f_t)(\sigma(t)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\quad + \sum_{t \in G} \left| \sum_{\sigma: E \cup G \xrightarrow{1 \dashrightarrow 1} H} \left( \prod_{i \in E, i \neq t} f_i(\sigma(i)) \right) (f_b f_t)(\sigma(t)) \prod_{j \in G} g_j(\sigma(j)) \right| \\
&\quad \text{(using induction on the quantities inside the absolute value signs} \\
&\quad \text{and viewing } f_b f_t \text{ as a single function)} \\
&\leq n((m+1) + (n-1))^{n-1} k^{\frac{n-1}{2} + m+1} \prod_{i \in F} \|f_i\|_\infty \prod_{j \in G} \|g_j\|_\infty \\
&\quad + m(m+n)^n k^{\frac{n}{2} + m} \prod_{i \in F} \|f_i\|_\infty \prod_{j \in G} \|g_j\|_\infty \\
&\leq (m+n+1)^{n+1} k^{\frac{n+1}{2} + m} \prod_{i \in F} \|f_i\|_\infty \prod_{j \in G} \|g_j\|_\infty.
\end{aligned}$$

□

Let  $\mathcal{U}_k^n$  denote the direct product of  $n$  copies of  $\mathcal{U}_k$ , and  $\mu_k^n$  denote the corresponding product measure. Let  $C(\mathcal{U}_k^n, \mathcal{M}_k(\mathbb{C}))$  denote the  $C^*$ -algebra of all continuous functions from  $\mathcal{U}_k^n$  into  $\mathcal{M}_k(\mathbb{C})$ . If  $\vec{u} = (u_1, \dots, u_n) \in \mathcal{U}_k^n$ , then the coordinate variables  $u_1, \dots, u_n$  are unitary elements of  $C(\mathcal{U}_k^n, \mathcal{M}_k(\mathbb{C}))$ .

The following lemma is a vastly improved estimate that is independent of the maximum cardinality of the indices in the integral. We require the elementary inequalities  $m^m \leq 2^{m^2}$  and  $\frac{1}{P(k,m)} \leq \frac{m^m}{k^m}$  for positive integers  $m \leq k$ .

**Lemma 5** *Suppose  $m$  is a positive integer. For every positive integers  $k, n$  with  $k \geq m$ , and for all subsets  $\{i_1, \dots, i_m\}$ ,  $\{j_1, \dots, j_m\}$  of  $\{1, \dots, k\}$ , and  $\{\iota_1, \dots, \iota_m, \eta_1, \dots, \eta_m\}$  of  $\{1, \dots, n\}$ ,*

$$\left| \int_{\mathcal{U}_k^n} f_{i_1 j_1}(u_{\iota_1}) \cdots f_{i_m j_m}(u_{\iota_m}) \overline{f_{s_1 t_1}(u_{\eta_1})} \cdots \overline{f_{s_m t_m}(u_{\eta_m})} d\mu_k^n(\vec{u}) \right| \leq \frac{4^{m^2}}{k^m}.$$

Proof. for  $1 \leq j \leq n$ , let  $T_j = \{1 \leq \lambda \leq m : \iota_\lambda = j\}$  and  $T_j^* = \{1 \leq \lambda \leq m : \eta_\lambda = j\}$ . Then

$$\begin{aligned} & \int_{\mathcal{U}_k^n} f_{i_1 j_1}(u_{\iota_1}) \cdots f_{i_m j_m}(u_{\iota_m}) \overline{f_{s_1 t_1}(u_{\eta_1})} \cdots \overline{f_{s_m t_m}(u_{\eta_m})} d\mu_k^n(\vec{u}) \\ &= \prod_{j=1}^n \int_{\mathcal{U}_k} \left( \prod_{\lambda \in T_j} f_{i_\lambda j_\lambda}(u_j) \prod_{\lambda \in T_j^*} \overline{f_{s_\lambda t_\lambda}(u_j)} \right) d\mu_k(u_j). \end{aligned}$$

Hence, we can assume that  $n = 1$ . Moreover, in view of the Cauchy-Schwarz inequality, it is sufficient to prove that

$$I = \int_{\mathcal{U}_k} |f_{i_1 j_1}(u)|^2 |f_{i_2 j_2}(u)|^2 \cdots |f_{i_m j_m}(u)|^2 d\mu_k(u) \leq \frac{4^{m^2}}{k^m}. \quad (1)$$

Let  $d$  be the maximum cardinality of the sets  $\{i_1, \dots, i_m\}$  and  $\{j_1, \dots, j_m\}$ . By replacing  $u$  with  $u^*$ , which does not alter the integral but interchanges  $i$ 's with  $j$ 's, we can assume that  $d$  is the cardinality of  $\{i_1, \dots, i_m\}$ . Then  $1 \leq d \leq m$ . Let  $B_{d,k}$  be the largest integral of the type in (1) with  $d = \text{Card}(\{i_1, \dots, i_m\})$ .

If  $d = m$ , then, by Lemma 3, the integral in (1) is at most  $\frac{1}{P(k,m)}$ , and  $\frac{1}{P(k,m)} \leq \frac{m^m}{k^m} \leq \frac{4^{m^2}}{k^m}$ .

Now we will prove that  $B_{d,k} \leq 2^m B_{d+1,k}$  whenever  $1 \leq d < m$ . For  $1 \leq d < m$ , assume that the integral  $I$  above equals  $B_{d,k}$ . Since  $d < m$ , at least two of  $i_1, \dots, i_m$  must be the same. From Lemma 1, we can assume that  $1 \leq i_1, \dots, i_m \leq d$  and  $1 = i_1 = i_2 = \dots = i_s$  and  $1 \notin \{i_{s+1}, \dots, i_m\}$ . Since  $k \geq m > d$ , we can define a unitary matrix  $v$  with 1 on the diagonal except in the  $(1, 1)$  and  $(k, k)$  positions, with  $\frac{1}{\sqrt{2}}$  in the  $(1, 1)$ ,  $(k, 1)$ ,  $(k, k)$  positions and  $-\frac{1}{\sqrt{2}}$  in the  $(1, k)$  position. Since the integral remains unchanged when we replace the variable  $u$  with  $vu$ , we obtain

$$\begin{aligned} B_{d,k} &= \frac{1}{2^s} \int_{\mathcal{U}_k} \prod_{\beta=1}^s |f_{1j_\beta}(u) + f_{kj_\beta}(u)|^2 \prod_{\alpha=s+1}^m |f_{i_\alpha j_\alpha}(u)|^2 d\mu_k(u) \\ &= \frac{1}{2^s} \int_{\mathcal{U}_k} \prod_{\beta=1}^s \left( |f_{1j_\beta}(u)|^2 + \overline{f_{1j_\beta}(u)} f_{kj_\beta}(u) + f_{1j_\beta}(u) \overline{f_{kj_\beta}(u)} + |f_{kj_\beta}(u)|^2 \right) d\mu_k(u). \end{aligned}$$

$$\begin{aligned}
& \prod_{\alpha=s+1}^m |f_{i_{\alpha}j_{\alpha}}(u)|^2 d\mu_k(u) \\
&= \frac{1}{2^s} \int_{\mathcal{U}_k} \prod_{\beta=1}^s |f_{1j_{\beta}}(u)|^2 \prod_{\alpha=s+1}^m |f_{i_{\alpha}j_{\alpha}}(u)|^2 d\mu_k(u) \\
&+ \frac{1}{2^s} \int_{\mathcal{U}_k} \prod_{\beta=1}^s |f_{kj_{\beta}}(u)|^2 \prod_{\alpha=s+1}^m |f_{i_{\alpha}j_{\alpha}}(u)|^2 d\mu_k(u) + \frac{1}{2^s} \int_{\mathcal{U}_k} \Delta d\mu_k(u),
\end{aligned}$$

where  $\Delta$  is a summation of  $4^s - 2$  terms with each of them having both an  $f_{1*}(u)$  and an  $f_{k*}(u)$  factor (with or without conjugation signs) and the maximum cardinality of the indices in each term is  $d + 1$ , which implies  $\left| \int_{\mathcal{U}_k} \Delta d\mu_k(u) \right| \leq (4^s - 2)B_{d+1,k}$ .

Since

$$\begin{aligned}
B_{d,k} &= \int_{\mathcal{U}_k} \prod_{\beta=1}^s |f_{1j_{\beta}}(u)|^2 \prod_{\alpha=s+1}^m |f_{i_{\alpha}j_{\alpha}}(u)|^2 d\mu_k(u) \\
&= \int_{\mathcal{U}_k} \prod_{\beta=1}^s |f_{kj_{\beta}}(u)|^2 \prod_{\alpha=s+1}^m |f_{i_{\alpha}j_{\alpha}}(u)|^2 d\mu_k(u),
\end{aligned}$$

we have

$$B_{d,k} \leq \frac{1}{2^s} (B_{d,k} + B_{d,k}) + \frac{1}{2^s} (4^s - 2) B_{d+1,k}.$$

Therefore

$$B_{d,k} \leq 2^m B_{d+1,k}.$$

It follows that  $B_{d,k} \leq 2^{m(m-d)} B_{m,k} \leq \frac{2^{m^2}}{P(k,m)} \leq \frac{2^{m^2} m^m}{k^m} \leq \frac{4^{m^2}}{k^m}$  when  $k \geq m$  and  $1 \leq d \leq m$ .  $\square$

For any positive integer  $m$ , let  $B(m)$  be the *Bell number* of  $m$ , i.e., the number of equivalence relations on a set with cardinality  $m$ . Suppose  $\mathcal{M}$  is a von Neumann algebra with a faithful tracial state  $\tau$  and  $\mathcal{U}(\mathcal{M})$  is the set of all unitary elements in  $\mathcal{M}$  and  $\vec{u} = (u_1, \dots, u_n) \in \mathcal{U}(\mathcal{M})^n$ . Let  $\mathbb{F}_n$  be a free group with standard generators  $h_1, \dots, h_n$ . Then there is a homomorphism  $\rho : \mathbb{F}_n \rightarrow \mathcal{U}(\mathcal{M})$  such that  $\rho(h_j) = u_j$ . We use the notation  $\rho(g) = g(\vec{u}) = g(u_1, \dots, u_n)$ .

D. Voiculescu [2] proved that a standard family of independent random unitary  $k \times k$  matrices and a constant  $k \times k$  unitary matrix is asymptotically free as  $k \rightarrow \infty$ . The following theorem gives a very elementary proof of a more detailed version of D. Voiculescu's result. The constants in the following theorem are far from best possible, but they are, at least, explicit.

**Theorem 6** *Suppose  $M > 0$  and  $m$  is a positive integer. For every reduced words  $g_1, \dots, g_w \in \mathbb{F}_n \setminus \{e\}$  with  $\sum_{i=1}^w \text{length}(g_i) = m$ , and commuting normal  $k \times k$  matrices  $x_1, \dots, x_w$  with trace 0 and  $\|x_i\| \leq M$  for all  $1 \leq i \leq w$ , we have*

1.

$$\left| \int_{\mathcal{U}_k^n} \tau_k(g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w) d\mu_k^n(\vec{u}) \right| \leq \frac{B(m) 2^{m^2} (Mw)^w}{k},$$

2.

$$\int_{\mathcal{U}_k^n} |\tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2 \cdots g_w(\vec{u})x_w)|^2 d\mu_k^n(\vec{u}) \leq \frac{B(2m)4^{m^2}(2Mw)^{2w}}{k^2},$$

3. if  $\varepsilon > 0$ , and  $k > \frac{2B(m)2^{m^2}(Mw)^w}{\varepsilon}$ , then

$$\mu_k^n(\{\vec{v} \in \mathcal{U}_k^n : |\tau_k(g_1(\vec{v})x_1g_2(\vec{v})x_2 \cdots g_w(\vec{v})x_w)| \geq \varepsilon\}) \leq \frac{4B(2m)4^{m^2}(2Mw)^{2w}}{k^2\varepsilon^2}.$$

Proof. Since  $x_1, \dots, x_w$  are commuting normal matrices, there is a unitary matrix  $v$  such that, for  $1 \leq j \leq w$ ,  $vx_jv^* = a_j$  is diagonal. Since  $\tau_k$  is tracial and

$$g_1(\vec{u})x_1g_2(\vec{u})x_2 \cdots g_w(\vec{u})x_w = v^*(g_1(v\vec{u}v^*)a_1g_2(v\vec{u}v^*)a_2 \cdots g_w(v\vec{u}v^*)a_w)v,$$

we have

$$\tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2 \cdots g_w(\vec{u})x_w) = \tau_k(g_1(v\vec{u}v^*)a_1g_2(v\vec{u}v^*)a_2 \cdots g_w(v\vec{u}v^*)a_w).$$

By the translation-invariance of  $\mu_k^n$ , we can assume that  $x_1, \dots, x_w$  are all diagonal matrices.

*Proof of the first statement.* Write  $g_1(\vec{u}) = u_{s_1}^{\varepsilon_1} \cdots u_{s_{m_1}}^{\varepsilon_{m_1}}$ ,  $g_2(\vec{u}) = u_{s_{m_1+1}}^{\varepsilon_{m_1+1}} \cdots u_{s_{m_2}}^{\varepsilon_{m_2}}$ ,  $\dots$ ,  $g_w(\vec{u}) = u_{s_{m_{w-1}+1}}^{\varepsilon_{m_{w-1}+1}} \cdots u_{s_{m_w}}^{\varepsilon_{m_w}}$  with each  $\varepsilon_j \in \{-1, 1\}$  and  $s_j \in \{1, \dots, n\}$  and with the property that  $s_j = s_{j+1}$  implies  $\varepsilon_j = \varepsilon_{j+1}$  unless  $j \in \{m_1, \dots, m_w\}$ . Note that  $m_w = m$  since  $\sum \text{length}(g_i) = m$ . Also write  $x_j = \text{diag}(\gamma_j(1), \dots, \gamma_j(k))$  for  $1 \leq j \leq w$ .

Define  $\dot{+}$  on  $\{1, \dots, m_w = m\}$  by  $s \dot{+} 1 = \begin{cases} 1, & s = m_w \\ s + 1, & 1 \leq s \leq m_w - 1 \end{cases}$ . Then we have

$$\begin{aligned} & \int_{\mathcal{U}_k^n} \tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2 \cdots g_w(\vec{u})x_w) d\mu_k^n(\vec{u}) \\ &= \frac{1}{k} \sum_{1 \leq i_1, \dots, i_{m_w \dot{+} 1} = i_1 \leq k} \left( \prod_{\nu=1}^w \gamma_\nu(i_{m_\nu \dot{+} 1}) \right) \int_{\mathcal{U}_k^n} \prod_{j=1}^{m_w} f_{i_j, i_{j+1}}(u_{s_j}^{\varepsilon_j}) d\mu_k^n(\vec{u}). \end{aligned}$$

Let  $E = \{1, 2, \dots, m_w\}$ . We can represent a choice of  $1 \leq i_1, \dots, i_{m_w} \leq k$  by a function  $\alpha : E \rightarrow H = \{1, \dots, k\}$ . Thus we can replace the sum  $\sum_{1 \leq i_1, \dots, i_{m_w \dot{+} 1} = i_1 \leq k}$  with  $\sum_{\alpha : E \rightarrow H}$  in the above equation. That is

$$(I) = \frac{1}{k} \sum_{\alpha : E \rightarrow H} \left( \prod_{\nu=1}^w \gamma_\nu(\alpha(m_\nu \dot{+} 1)) \right) \int_{\mathcal{U}_k^n} \prod_{j=1}^{m_w} f_{\alpha(j), \alpha(j+1)}(u_{s_j}^{\varepsilon_j}) d\mu_k^n(\vec{u}).$$

We only need to restrict sums to the functions  $\alpha$  such that the integral

$$I(\alpha) = \int_{\mathcal{U}_k^n} \prod_{j=1}^{m_w} f_{\alpha(j), \alpha(j+1)}(u_{s_j}^{\varepsilon_j}) d\mu_k^n(\vec{u}) \neq 0.$$

We call such function  $\alpha$  *good*, thus

$$I = \frac{1}{k} \sum_{\substack{\alpha : E \rightarrow H \\ \alpha \text{ is good}}} \left( \prod_{\nu=1}^w \gamma_{\nu}(\alpha(m_{\nu}+1)) \right) I(\alpha).$$

Lemma 2 tells us that  $m_w$  must be even and exactly half of the  $\varepsilon_j$ 's are 1 and the other half are  $-1$ . We know from Lemma 5 that

$$|I(\alpha)| \leq \frac{4^{(m/2)^2}}{k^{m/2}} \leq \frac{2^{m^2}}{k^{m/2}}. \quad (2)$$

Moreover, Lemma 2 says that if  $j \in E$  but  $j \notin \{1+m_1, \dots, 1+m_w\}$ , then  $\alpha(j) = \alpha(j')$  for some  $j' \neq j$ .

Next we define an equivalence relation  $\sim_{\alpha}$  on  $E$  by saying  $i \sim_{\alpha} j$  if and only if  $\alpha(i) = \alpha(j)$ . Note that if  $\beta : E \rightarrow H$ , then the relations  $\sim_{\alpha}$  and  $\sim_{\beta}$  are equal if and only if there is a permutation  $\sigma : H \rightarrow H$  such that  $\beta = \sigma \circ \alpha$ . We define an equivalent relation  $\approx$  on the set of all good functions by

$$\alpha \approx \beta \text{ if and only if } \sim_{\alpha} = \sim_{\beta}.$$

It is clear that

$$\alpha \approx \beta \implies I(\alpha) = I(\beta).$$

If  $j \in E$ , let  $[j]_{\alpha}$  denote the  $\sim_{\alpha}$ -equivalence class of  $j$ , and let  $E_{\alpha}$  denote the set of all such equivalence classes. We can construct all of the functions  $\beta$  equivalent to  $\alpha$  in terms of injective functions

$$\sigma : E_{\alpha} \xrightarrow{1-1} H$$

by defining

$$\beta(j) = \sigma([j]_{\alpha}).$$

Let  $A$  contains exactly one function  $\alpha$  from each  $\approx$ -equivalence class of good functions. Then we can write

$$\begin{aligned} |I| &= \left| \frac{1}{k} \sum_{\substack{\alpha : E \rightarrow H \\ \alpha \text{ is good}}} \left( \prod_{\nu=1}^w \gamma_{\nu}(\alpha(m_{\nu}+1)) \right) I(\alpha) \right| \\ &= \left| \frac{1}{k} \sum_{\alpha \in A} I(\alpha) \sum_{\beta \approx \alpha} \prod_{\nu=1}^w \gamma_{\nu}(\beta(m_{\nu}+1)) \right| \\ &= \frac{1}{k} \left| \sum_{\alpha \in A} I(\alpha) \sum_{\sigma : E_{\alpha} \xrightarrow{1-1} H} \prod_{\nu=1}^w \gamma_{\nu}(\sigma([m_{\nu}+1]_{\alpha})) \right| \\ &\leq \frac{1}{k} \sum_{\alpha \in A} |I(\alpha)| \left| \sum_{\sigma : E_{\alpha} \xrightarrow{1-1} H} \prod_{\nu=1}^w \gamma_{\nu}(\sigma([m_{\nu}+1]_{\alpha})) \right|. \end{aligned} \quad (3)$$

Also we know that

$$\text{Card}(A) \leq B(m). \quad (4)$$

We only need to focus on  $\left| \sum_{\sigma: E_\alpha \xrightarrow{1-1} H} \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right|$ . Let

$$F_\alpha = \{ [m_\nu \dot{+} 1]_\alpha : 1 \leq \nu \leq w, \text{Card}([m_\nu \dot{+} 1]_\alpha) = 1 \},$$

$$G_\alpha = \{ [m_\nu \dot{+} 1]_\alpha : 1 \leq \nu \leq w, \text{Card}([m_\nu \dot{+} 1]_\alpha) > 1 \},$$

$$K_\alpha = E_\alpha \setminus (F_\alpha \cup G_\alpha).$$

Since the product  $\prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha))$  is determined once  $\sigma$  is defined on  $F_\alpha \cup G_\alpha$ , it follows that this product is repeated at most  $P(k, \text{card}(K_\alpha))$  times. Hence we have

$$\begin{aligned} & \left| \sum_{\sigma: E_\alpha \xrightarrow{1-1} H} \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right| \\ & \leq P(k, \text{card}(K_\alpha)) \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} F} \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right| \\ & \leq k^{\text{card}(K_\alpha)} \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} F} \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right|. \end{aligned} \quad (5)$$

If  $a = [m_\nu \dot{+} 1]_\alpha \in F_\alpha$ , from the definition of  $F_\alpha$ , it is clear that  $\nu$  is unique. Then define  $f_a(\sigma(a)) = \gamma_\nu(\sigma(a))$ . By  $\tau_k(x_i) = 0$  for all  $1 \leq i \leq w$ , it follows that  $\sum_{s=1}^k f_a(s) = 0$ . If  $b = [m_\nu \dot{+} 1]_\alpha \in G_\alpha$ , from the definition of  $G_\alpha$ , the cardinality of  $b$  is greater than 1, say it  $r$ . Then define  $g_b(\sigma(b)) = (\gamma_\nu(\sigma(b)))^r$ . Therefore

$$\begin{aligned} & \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} H} \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right| \\ & = \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} H} \prod_{a \in F_\alpha} f_a(\sigma(a)) \prod_{b \in G_\alpha} g_b(\sigma(b)) \right| \\ & \quad (\text{letting } F = F_\alpha, G = G_\alpha \text{ and using Lemma 4}) \\ & \leq k^{[\text{card}(F_\alpha)/2] + \text{card}(G_\alpha)} w^w M^w. \end{aligned} \quad (6)$$

As we mentioned before that  $\text{card}([j]_\alpha) = 1$  implies  $[j]_\alpha \in F_\alpha$ , we see that

$$[\text{card}(F_\alpha)/2] + \text{card}(G_\alpha) + \text{card}(K_\alpha) \leq \text{card}(E)/2 = m_w/2. \quad (7)$$

Combining (2), (3), (4), (5), (6) and (7) together, we have

$$|I| \leq \frac{1}{k} B(m) 2^{m^2} (Mw)^w.$$



*Proof of the second statement.* We know that

$$\begin{aligned}
& |\tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w)|^2 \\
&= \tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w) \cdot \overline{\tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w)} \\
&= \frac{1}{k^2} \sum_{1 \leq i_1, \dots, i_{m_w+1} = i_1 \leq k} \left( \prod_{\nu=1}^w \gamma_\nu(i_{m_\nu+1}) \right) \prod_{j=1}^{m_w} f_{i_j i_{j+1}}(u_{s_j}^{\varepsilon_j}) \\
&\quad \sum_{1 \leq l_1, \dots, l_{m_w+1} = l_1 \leq k} \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(l_{m_\lambda+1})} \right) \prod_{t=1}^{m_w} \overline{f_{l_t l_{t+1}}(u_{s_t}^{\varepsilon_t})}.
\end{aligned}$$

Define  $\dot{+}$  on the set  $\{1, 2, \dots, 2m_w\}$  by  $x \dot{+} 1 = \begin{cases} 1, & x = m_w \\ m_w + 1, & x = 2m_w \\ x + 1, & 1 \leq x \leq m_w - 1 \end{cases}$ . Then we have

$$\begin{aligned}
I &= \int_{\mathcal{U}_k^n} |\tau_k(g_1(\vec{u})x_1g_2(\vec{u})x_2\cdots g_w(\vec{u})x_w)|^2 d\mu_k^n(\vec{u}) \\
&= \frac{1}{k^2} \int_{\mathcal{U}_k^n} \sum_{1 \leq i_1, \dots, i_{m_w+1} \leq k} \left( \prod_{\nu=1}^w \gamma_\nu(i_{m_\nu+1}) \right) \prod_{j=1}^{m_w} f_{i_j i_{j+1}}(u_{s_j}^{\varepsilon_j}) \\
&\quad \sum_{1 \leq l_1, \dots, l_{m_w+1} \leq k} \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(l_{m_\lambda+1})} \right) \prod_{t=1}^{m_w} \overline{f_{l_t l_{t+1}}(u_{s_t}^{\varepsilon_t})}.
\end{aligned}$$

Let  $E = \{1, 2, \dots, 2m_w\}$ . We can represent a choice of  $1 \leq i_1, \dots, i_{m_w} \leq k$  by a function  $\alpha : E \rightarrow H = \{1, \dots, k\}$ . Thus we can rewrite  $I$

$$\begin{aligned}
I &= \frac{1}{k^2} \sum_{\alpha: E \rightarrow H} \left( \prod_{\nu=1}^w \gamma_\nu(\alpha(m_\nu \dot{+} 1)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\alpha((m_\lambda \dot{+} 1) + m_w))} \right) \\
&\quad \int_{\mathcal{U}_k^n} \prod_{j=1}^{m_w} f_{\alpha(j)\alpha(j \dot{+} 1)}(u_{s_j}^{\varepsilon_j}) \prod_{t=1}^{m_w} \overline{f_{\alpha(t+m_w)\alpha((t \dot{+} 1) + m_w)}(u_{s_t}^{\varepsilon_t})}.
\end{aligned}$$

We only need to restrict sums to the functions  $\alpha$  such that the integral

$$I(\alpha) = \int_{\mathcal{U}_k^n} \prod_{j=1}^{m_w} f_{\alpha(j)\alpha(j \dot{+} 1)}(u_{s_j}^{\varepsilon_j}) \prod_{t=1}^{m_w} \overline{f_{\alpha(t+m_w)\alpha((t \dot{+} 1) + m_w)}(u_{s_t}^{\varepsilon_t})} \neq 0.$$

We call such function  $\alpha$  *good*. We have

$$I = \frac{1}{k^2} \sum_{\alpha: E \rightarrow H} \left( \prod_{\nu=1}^w \gamma_\nu(\alpha(m_\nu \dot{+} 1)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\alpha((m_\lambda \dot{+} 1) + m_w))} \right) I(\alpha). \quad (8)$$

Lemma 2 tells us that if  $j \in E$  but  $j \notin \{1 \dot{+} m_1, \dots, 1 \dot{+} m_w, (1 \dot{+} m_1) + m_w, \dots, (1 \dot{+} m_w) + m_w\}$ , then  $\alpha(j) = \alpha(j')$  for some  $j' \neq j$ . We know from Lemma 5 that

$$|I(\alpha)| \leq \frac{4(m)^2}{k^m}. \quad (9)$$

Next we define an equivalence relation  $\sim_\alpha$  on  $E$  by saying  $i \sim_\alpha j$  if and only if  $\alpha(i) = \alpha(j)$ . Note that if  $\beta : E \rightarrow H$ , then the relations  $\sim_\alpha$  and  $\sim_\beta$  are equal if and only if there is a permutation  $\sigma : H \rightarrow H$  such that  $\beta = \sigma \circ \alpha$ . We define an equivalent relation  $\approx$  on the set of all good functions by

$$\alpha \approx \beta \text{ if and only if } \sim_\alpha = \sim_\beta.$$

It is clear that

$$\alpha \approx \beta \implies I(\alpha) = I(\beta).$$

If  $j \in E$ , let  $[j]_\alpha$  denote the  $\sim_\alpha$ -equivalence class of  $j$ , and let  $E_\alpha$  denote the set of all such equivalence classes. We can easily construct all of the functions  $\beta$  equivalent to  $\alpha$  in terms of injective functions

$$\sigma : E_\alpha \xrightarrow{1-1} H$$

by defining

$$\beta(j) = \sigma([j]_\alpha).$$

Let  $A$  contains exactly one function  $\alpha$  from each  $\approx$ -equivalence class of good functions. Then we can write

$$\begin{aligned} I &= |I| = \left| \frac{1}{k^2} \sum_{\alpha: E \rightarrow H} \left( \prod_{\nu=1}^w \gamma_\nu(\alpha(m_\nu \dot{+} 1)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\alpha((m_\lambda \dot{+} 1) + m_w))} \right) I(\alpha) \right| \\ &= \left| \frac{1}{k^2} \sum_{\alpha \in A} I(\alpha) \sum_{\beta \approx \alpha} \left( \prod_{\nu=1}^w \gamma_\nu(\beta(m_\nu \dot{+} 1)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\beta((m_\lambda \dot{+} 1) + m_w))} \right) \right| \\ &= \frac{1}{k^2} \left| \sum_{\alpha \in A} I(\alpha) \sum_{\sigma: E_\alpha \rightarrow H} \left( \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\sigma([ (m_\lambda \dot{+} 1) + m_w ]_\alpha))} \right) \right| \\ &\leq \frac{1}{k^2} \sum_{\alpha \in A} |I(\alpha)| \left| \sum_{\sigma: E_\alpha \xrightarrow{1-1} H} \left( \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\sigma([ (m_\lambda \dot{+} 1) + m_w ]_\alpha))} \right) \right|. \end{aligned} \quad (10)$$

We know that

$$\text{Card}(A) \leq B(2m). \quad (11)$$

We only need to focus on

$$\left| \sum_{\sigma: E_\alpha \xrightarrow{1-1} H} \left( \prod_{\nu=1}^w \gamma_\nu(\sigma([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda(\sigma([ (m_\lambda \dot{+} 1) + m_w ]_\alpha))} \right) \right|.$$

Let

$$\begin{aligned} F_\alpha^1 &= \{ [m_\nu \dot{+} 1]_\alpha : 1 \leq \nu \leq w, \text{Card}([m_\nu \dot{+} 1]_\alpha) = 1 \}, \\ F_\alpha^2 &= \{ [(m_\nu \dot{+} 1) + m_w]_\alpha : 1 \leq \nu \leq w, \text{Card}([(m_\nu \dot{+} 1) + m_w]_\alpha) = 1 \}, \\ G_\alpha^1 &= \{ [m_\nu \dot{+} 1]_\alpha : 1 \leq \nu \leq w, \text{Card}([m_\nu \dot{+} 1]_\alpha) > 1 \}, \\ G_\alpha^2 &= \{ [(m_\nu \dot{+} 1) + m_w]_\alpha : 1 \leq \nu \leq w, \text{Card}([(m_\nu \dot{+} 1) + m_w]_\alpha) > 1 \}, \end{aligned}$$

$$K_\alpha = E_\alpha \setminus (F_\alpha^1 \cup F_\alpha^2 \cup G_\alpha^1 \cup G_\alpha^2).$$

Since the product  $\prod_{\nu=1}^w \gamma_\nu (\sigma ([m_\nu \dot{+} 1]_\alpha))$  is determined once  $\sigma$  is defined on  $F_\alpha^1 \cup F_\alpha^2 \cup G_\alpha^1 \cup G_\alpha^2$ , it follows that this product is repeated at most  $P(k, \text{card}(K_\alpha))$  times. Hence we have

$$\begin{aligned} & \left| \sum_{\sigma: E_\alpha \xrightarrow{1-1} H} \left( \prod_{\nu=1}^w \gamma_\nu (\sigma ([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda (\sigma ([m_\lambda \dot{+} 1] + m_w)_\alpha)} \right) \right| \\ & \leq P(k, \text{card}(K_\alpha)) \left| \sum_{\sigma: F_\alpha^1 \cup F_\alpha^2 \cup G_\alpha^1 \cup G_\alpha^2 \xrightarrow{1-1} F} \left( \prod_{\nu=1}^w \gamma_\nu (\sigma ([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda (\sigma ([m_\lambda \dot{+} 1] + m_w)_\alpha)} \right) \right| \\ & \leq k^{\text{card}(K_\alpha)} \left| \sum_{\sigma: F_\alpha^1 \cup F_\alpha^2 \cup G_\alpha^1 \cup G_\alpha^2 \xrightarrow{1-1} F} \left( \prod_{\nu=1}^w \gamma_\nu (\sigma ([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda (\sigma ([m_\lambda \dot{+} 1] + m_w)_\alpha)} \right) \right|. \quad (12) \end{aligned}$$

If  $a \in F_\alpha^1$  (or  $F_\alpha^2$ ), from the definition of  $F_\alpha^1$  (or  $F_\alpha^2$ ), the cardinality of  $a$  is 1. Then define  $f_a(\sigma(a)) = \gamma_\nu(\sigma(a))$  (or  $\overline{\gamma_\nu(\sigma(a))}$ ). By  $\tau_k(x_i) = 0$  for all  $1 \leq i \leq w$ , it follows that  $\sum_{s=1}^k f_a(s) = 0$ . If  $b \in G_\alpha^1$  (or  $G_\alpha^2$ ), from the definition of  $G_\alpha^1$  (or  $G_\alpha^2$ ), the cardinality of  $b$  is greater than 1, say it  $r$ . Then define  $g_b(\sigma(b)) = (\gamma_\nu(\sigma(b)))^r$  (or  $(\overline{\gamma_\nu(\sigma(b))})^r$ ). Let  $F_\alpha = F_\alpha^1 \cup F_\alpha^2$  and  $G_\alpha = G_\alpha^1 \cup G_\alpha^2$ . Then we have

$$\begin{aligned} & \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} F} \left( \prod_{\nu=1}^w \gamma_\nu (\sigma ([m_\nu \dot{+} 1]_\alpha)) \right) \left( \prod_{\lambda=1}^w \overline{\gamma_\lambda (\sigma ([m_\lambda \dot{+} 1] + m_w)_\alpha)} \right) \right| \\ & = \left| \sum_{\sigma: F_\alpha \cup G_\alpha \xrightarrow{1-1} H} \prod_{a \in F_\alpha} f_a(\sigma(a)) \prod_{b \in G_\alpha} g_b(\sigma(b)) \right| \\ & \quad (\text{letting } F = F_\alpha, G = G_\alpha \text{ and using Lemma 4}) \\ & \leq k^{[\text{card}(F_\alpha)/2] + \text{card}(G_\alpha)} (2w)^{2w} M^{2w}. \quad (13) \end{aligned}$$

As we mentioned before that  $\text{card}([j]_\alpha) = 1$  implies  $[j]_\alpha \in F_\alpha$ , we see that

$$[\text{card}(F_\alpha)/2] + \text{card}(G_\alpha) + \text{card}(K_\alpha) \leq \text{card}(E)/2 = 2m_w/2 = m. \quad (14)$$

Combining (9), (10), (11), (12), (13) and (14) together, we have

$$|I| \leq \frac{1}{k^2} B(2m) 4^{m^2} (2Mw)^{2w}.$$

This completes the proof of the second statement.

*Proof of the third statement.* The third statement follows from statement 1 and statement 2 and Chebychev's inequality.

Let  $A = B(m) 2^{m^2} (Mw)^w$  and  $B = B(2m) 4^{m^2} (2Mw)^{2w}$ . Define

$$f(\vec{v}) = \tau_k(g_1(\vec{u}) x_1 g_2(\vec{u}) x_2 \cdots g_w(\vec{u}) x_w).$$

Then the expected value of  $f$   $E(f) = \int_{\mathcal{U}_k^n} f(\vec{v}) d\mu_k^n(\vec{v})$ , and the variance of  $f$  is

$$\begin{aligned} \text{Var}(f) &= \int_{\mathcal{U}_k^n} |\tau_k(g_1(\vec{v})x_1g_2(\vec{v})x_2 \cdots g_w(\vec{v})x_w)|^2 d\mu_k^n(\vec{v}) \\ &\quad - \left| \int_{\mathcal{U}_k^n} \tau_k(g_1(\vec{v})x_1g_2(\vec{v})x_2 \cdots g_w(\vec{v})x_w) d\mu_k^n(\vec{v}) \right|^2 \\ &\leq \int_{\mathcal{U}_k^n} |\tau_k(g_1(\vec{v})x_1g_2(\vec{v})x_2 \cdots g_w(\vec{v})x_w)|^2 d\mu_k^n(\vec{v}) \\ &\leq \frac{B}{k^2}. \end{aligned}$$

Since

$$|f(\vec{v}) - E(f)| \geq |f(\vec{v})| - |E(f)| \geq \varepsilon - \frac{A}{k} > \frac{\varepsilon}{2},$$

we have  $\{\vec{v} : |f(\vec{v})| \geq \varepsilon\} \subseteq \{\vec{v} : |f(\vec{v}) - E(f)| \geq \frac{\varepsilon}{2}\}$ . Therefore from Chebychev's inequality, we have

$$\mu_k^n(\{\vec{v} : |f(\vec{v})| \geq \varepsilon\}) \leq \frac{\text{Var}(f)}{\varepsilon^2} \leq \frac{4B}{k^2\varepsilon^2}.$$

□

The following corollary is a direct consequence of the third statement of Theorem 6.

**Corollary 7** *Suppose  $M, N, k$  are positive integers. Let  $\mathcal{D}$  be a finite set of commuting normal matrices with trace 0 in  $\mathcal{M}_k(\mathbb{C})$  and  $\|x\| \leq M$  for all  $x \in \mathcal{D}$ . Let*

$$\begin{aligned} \mathcal{E} &= \{(g_1, \dots, g_r, x_1, \dots, x_r) : r \in \mathbb{N}, g_1, \dots, g_r \text{ are reduced words in } \mathbb{F}_n \setminus \{e\} \\ &\quad \text{such that } \sum_{i=1}^r \text{length}(g_i) \leq N, \text{ and } x_1, \dots, x_r \in \mathcal{D}\}. \end{aligned}$$

Suppose  $\mathfrak{e} = (g_1, \dots, g_r, x_1, \dots, x_r) \in \mathcal{E}$ , define  $\mathfrak{e}(\vec{v}) = g_1(\vec{v})x_1 \cdots g_r(\vec{v})x_r$ . Then

$$\mu_k^n \left( \bigcap_{\mathfrak{e} \in \mathcal{E}} \{\vec{v} : |\tau_k(\mathfrak{e}(\vec{v}))| < \varepsilon\} \right) \geq 1 - \text{card}(\mathcal{E}) \frac{4B(2m)4^{m^2}(2Mw)^{2w}}{k^2\varepsilon^2}.$$

Lemma 5.1 [3] follows directly from the corollary above.

Let  $\mathcal{M}$  be a von Neumann algebra with a tracial state  $\tau$  and  $X_1, X_2, \dots, X_n$  be elements in  $\mathcal{M}$ . For any  $R, \varepsilon > 0$ , and positive integers  $m$  and  $k$ , define  $\Gamma_R(X_1, \dots, X_n; m, k, \varepsilon)$  to be the subset of  $\mathcal{M}_k(\mathbb{C})^n$  consisting of all  $(x_1, \dots, x_n)$  in  $\mathcal{M}_k(\mathbb{C})^n$  such that  $\|x_j\| \leq R$  for  $1 \leq j \leq n$ , and

$$|\tau_k(x_{i_1}^{\eta_1} \cdots x_{i_q}^{\eta_q}) - \tau(X_{i_1}^{\eta_1} \cdots X_{i_q}^{\eta_q})| < \varepsilon,$$

for all  $1 \leq i_1, \dots, i_q \leq n$ , all  $\eta_1, \dots, \eta_q \in \{1, *\}$  and all  $q$  with  $1 \leq q \leq m$ .

Suppose  $\vec{U}$  is a  $n$ -tuple in  $\mathcal{M}$  and, for each positive integer  $k$ ,  $\vec{u}_k$  is a  $n$ -tuple in  $\mathcal{M}_k(\mathbb{C})$ , then we say  $\vec{u}_k$  converges to  $\vec{U}$  in distribution if  $p(\vec{u}_k) \rightarrow p(\vec{U})$  for all  $*$ -monomials  $p$ .

**Corollary 8** *Let  $M, m$  be positive integers and  $\varepsilon > 0$ . Suppose  $\mathcal{M}$  is a von Neumann algebra with a faithful trace  $\tau$ . Suppose  $X_1, \dots, X_s$  are commuting normal operators in  $\mathcal{M}$ ,  $U_1, \dots, U_n$  are free Haar unitary elements in  $\mathcal{M}$  and  $\{X_1, \dots, X_s\}, \{U_1, \dots, U_n\}$  are free. For any positive integer  $k$ , let  $\{x(k, 1), \dots, x(k, s)\}$  be a set of commuting normal  $k \times k$  matrices such that  $\sup_{k,j} \|x(k, j)\| \leq M$  and*

$$(x(k, 1), \dots, x(k, s)) \rightarrow (X_1, \dots, X_s)$$

*in distribution as  $k \rightarrow \infty$ .*

*If*

$$\Omega_k = \{(v_1, \dots, v_n) \in \mathcal{U}_k^n : (x(k, 1), \dots, x(k, s), v_1, \dots, v_n) \in \Gamma_1(X_1, \dots, X_s, U_1, \dots, U_n; m, k, \varepsilon)\},$$

*then*

$$\lim_{k \rightarrow \infty} \mu_k^n(\Omega_k) = 1.$$

Lemma 5.2 [3] follows directly from the corollary above.

We end this paper with one last corollary.

**Corollary 9** *Let  $M, m$  be positive integers and  $\varepsilon > 0$ . Suppose  $\mathcal{M}$  is a von Neumann algebra with a faithful trace  $\tau$ . Suppose  $X_1, \dots, X_s$  are free normal operators in  $\mathcal{M}$ . Suppose  $\{x(k, 1), \dots, x(k, s)\}$  is a set of normal  $k \times k$  matrices such that  $\sup_{k,j} \|u(k, j)\| \leq M$  and, for  $1 \leq j \leq s$ ,  $x(k, j) \rightarrow X_j$  in distribution as  $k \rightarrow \infty$ .*

*If*

$$\Theta_k = \{(v_1, \dots, v_s) \in \mathcal{U}_k^s : (v_1^* x(k, 1) v_1, \dots, v_s^* x(k, s) v_s) \in \Gamma_1(X_1, \dots, X_s; m, k, \varepsilon)\},$$

*then*

$$\lim_{k \rightarrow \infty} \mu_k^n(\Theta_k) = 1.$$

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